



Multiple Splitting for Parabolic Equations in Almost Periodic Spaces

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Abstract. In this paper, we study the initial value problem for parabolic non-autonomous equations with multiple nonlinearities. We develop a splitting method that allows us to analyze equations with several non-linear terms. We use the method to prove well-posedness of the fractional one-dimensional nonlinear heat equation, with a potential term in almost periodic spaces.

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1. Introduction

In this paper, we consider non-autonomous parabolic equations of the form,

$$\partial_t u + Au = \sum_{j=1}^n F_j(t, x, u), \quad (1.1)$$

where $u(t, x) \in X$ for $x \in \mathbb{R}^n$, $t > 0$ and $F_j : \mathbb{R} \times X \rightarrow X$ a continuous map, A is a infinitesimal generator of a strongly continuous semigroup of operators and X a Banach space. We consider the initial value problem $u(x, 0) = u_0(x)$. Equation (1.1) arises in several mathematical applications, mostly those that are modelled by reaction-diffusion equations [9, 10, 15] and Complex Ginzburg–Landau equations [1]. High order splitting techniques were developed based on Lie–Trotter schemes for the numerical approximation of solutions [8, 13]. Recently, these splitting methods were used to study well-posedness of reaction–diffusion equation [5, 6] and Ginzburg–Landau equation [3, 4]. The advantage of using splitting methods is that allow us to prove results concerning the main equation analyzing the linear and non-linear parts separately. This exchanges a difficult problem into multiple simpler ones. The aim of this paper is to improve the splitting method with multiple non-linear terms.

As an example, we study a fractional heat equation with a potential term in a subset of almost periodic functions. Heat equations with potential terms were studied with different characteristics [2, 16]. Almost periodic functions were introduced and developed by Bohr [7], and resembles the Fourier theory. These are well studied for reaction-diffusion equations [17]. We prove that, if the initial condition, has specific almost periodic phases, then the solution of the fractional heat equation with a potential term, conserves the same phases.

This paper is organized as follows:

- Section 2 We introduce the notation and prove preliminary results concerning the linear and non-linear components of the main parabolic equation.
- Section 3 We introduce the propagators, allowing us to construct a splitting approximate equation. This is important to build up the linear part.
- Section 4 We obtain several results to finally prove the convergence of the “splitting” equation to the “original” equation. This allows us to study separately, the linear and non-linear parts, to obtain interesting results about the original equation.
- Section 5 We expose an example, concerning a one-dimensional fractional heat equation with a potential term in a subset of almost periodic functions.

2. Notations and Preliminaries

We present our results in the more general space possible. In this sense, we work with Banach spaces and for the applications in continuous functions with vectorial values, that is to say, Banach space valued functions.

Let Z be a Banach space, we define $C_u(\mathbb{R}^d, Z)$ as the set of uniformly continuous and bounded functions on $\mathbb{R}^d$ with values in Z . Defining the norm

$$\|u\|_{\infty, Z} = \sup_{x \in \mathbb{R}^d} |u(x)|_Z,$$

$C_u(\mathbb{R}^d, Z)$ is a Banach space. It is easy to see that if $g \in L^1(\mathbb{R}^d)$ and $u \in C_u(\mathbb{R}^d, Z)$ the Bochner integral is defined in the following way,

$$(g * u)(x) = \int_{\mathbb{R}^d} g(y)u(x-y)dy$$

This defines an element of $C_u(\mathbb{R}^d, Z)$ and the linear operator $u \mapsto g * u$ is continuous (see [12]).

We consider integral solutions of the problem (1.1). If $S(t)$ is a strongly continuous semigroup of operators then we say that $u \in C([0, T], C_u(\mathbb{R}^d, Z))$ is a mild solution of (1.1) if u verifies

$$u(t) = S(t)u_0 + \int_0^t S(t-t')F(t', x, u(t'))dt'. \quad (2.2)$$

where $F(t, x, u(t)) = \sum_{j=1}^n F_j(t, x, u)$. Since our method to build solutions of (2.2) is based on the Lie-Trotter method, it is necessary to study the non-linear problem related to F . We remark that some regularity condition is necessary for convergence, as it is shown in the counterexample given in [11].

Let $F : \mathbb{R}_+ \times \mathbb{R}^d \times Z \rightarrow Z$ be a continuous map, we say that is locally Lipschitz if, given $R, T > 0$ there exists $L = L(R, T) > 0$ such that if $t \in [0, T]$ and $z, \tilde{z} \in Z$ with $|z|_Z, |\tilde{z}|_Z \leq R$, then

$$|F(t, x, z) - F(t, x, \tilde{z})|_Z \leq L|z - \tilde{z}|_Z.$$

In this case, for any $t_0 \in \mathbb{R}^+$, $z_0 \in Z$ there exists a unique (maximal) solution of the Cauchy problem

$$z(t) = z_0 + \int_{t_0}^t F(t', x, z(t')) dt' \quad (2.3)$$

defined on $[t_0, t_0 + T^*(t_0, z_0))$, with $T^*(t_0, z_0)$ the maximal time of existence. It is easy to see that there exists a nonincreasing function $\mathcal{T} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, such that

$$\mathcal{T}(T, R) \leq \inf\{T^*(t_0, z_0) : 0 \leq t_0 \leq T, |z_0|_Z \leq R\}.$$

Also, one of the following alternatives holds:

- $T^*(t_0, z_0) = \infty$;
- $T^*(t_0, z_0) < \infty$ and $|z(t)|_Z \rightarrow \infty$ when $t \uparrow t_0 + T^*(t_0, z_0)$.

We can see that $F : \mathbb{R}_+ \times \mathbb{R}^d \times C_u(\mathbb{R}^d, Z) \rightarrow C_u(\mathbb{R}^d, Z)$, given by $F(t, x, u)(x) = F(t, x, u(x))$ is continuous and locally Lipschitz. Therefore, we can consider problem (2.3) in $C_u(\mathbb{R}^d, Z)$. We denote by $\mathbf{N} : \mathbb{R} \times \mathbb{R} \times C_u(\mathbb{R}^d, Z) \rightarrow C_u(\mathbb{R}^d, Z)$ the flow generated by the integral equation (2.3) as $u(t) = \mathbf{N}(t, t_0, u_0)$, defined for $t_0 \leq t < t_0 + T^*(t_0, u_0)$.

Theorem 2.1. *There exists a function $T^* : C_u(\mathbb{R}^d, Z) \rightarrow \mathbb{R}_+$ such that for $u_0 \in C_u(\mathbb{R}^d, Z)$, exists a unique $u \in C([0, T^*(u_0)), C_u(\mathbb{R}^d, Z))$ mild solution of (1.1) with $u(0) = u_0$. Moreover, one of the following alternatives holds:*

- $T^*(u_0) = \infty$;
- $T^*(u_0) < \infty$ and $\lim_{t \uparrow T^*(u_0)} \|u(t)\|_{\infty, Z} = \infty$.

Proof. See Theorem 4.3.4 in [12]. □

Proposition 2.1. *Under conditions of theorem above, then*

1. $T^* : C_u(\mathbb{R}^d, Z) \rightarrow \mathbb{R}_+$ is lower semi-continuous;
2. If $u_{0,n} \rightarrow u_0$ in $C_u(\mathbb{R}^d, Z)$ and $0 < T < T^*(u_0)$, then $u_n \rightarrow u$ in the Banach space $C([0, T], C_u(\mathbb{R}^d, Z))$.

Proof. See Proposition 4.3.7 in [12]. □

3. Propagators

To build the approximate solutions, we decompose the time variable in regular intervals and consider the evolution, in an alternate form, of the linear and

non-linear problem. To achieve this, we *turn on and off* each term of the equation. The first step is to consider the abstract linear problem,

$$\begin{aligned}\partial_t u - \alpha_0(t)Au &= 0, \\ u(s) &= u_0,\end{aligned}$$

where $\alpha_0(t) \geq 0$ and A is the infinitesimal generator of S , a strongly continuous semigroup of operators defined in the Banach space X . The mild solution of the non-autonomous problem can be written as $u(t) = S_{\alpha_0}(t, s)u_0 = S(\tau(t, s))u_0$, where τ is defined by

$$\tau(t, s) = \int_s^t \alpha_0(t') dt'$$

Formally, we have $\partial_t u = \partial_t S(\tau(t, s))u_0 = \partial_t \tau(t, s)AS(\tau(t, s))u_0$. To analyze the Lie-Trotter method, we define $\alpha_0 : \mathbb{R} \rightarrow \mathbb{R}$ a periodic function as:

$$\alpha_0(t) = \begin{cases} 1/b_0, & \text{if } k \leq t \leq k + b_0, \\ 0, & \text{if } k + b_0 \leq t \leq k + 1, \end{cases} \quad (3.4)$$

for $k \in \mathbb{Z}$. Given $h > 0$, we define the function $\alpha_{0h} : \mathbb{R} \rightarrow \mathbb{R}$ as $\alpha_{0h}(t) = \alpha_0(t/h)$. Clearly $0 \leq \alpha_{0h} \leq 1/b_0$ and α_{0h} is h -periodic. We consider $\tau_h : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$\tau_h(t, t') = \int_{t'}^t \alpha_{0h}(t'') dt'',$$

The following results show that $S_{\alpha_{0h}} = S_h$ defines a propagator in X . We also obtain some estimates that we will use in the following section concerning the convergence. We can prove that

Lemma 3.1. *The map τ_h is continuous in $\mathbb{R}^2$ and satisfies*

- i. $0 \leq \tau_h(t, t') \leq (1/b_0)(t - t')$, if $t' \leq t$,
- ii. $\tau_h(t, t') + \tau_h(t', t'') = \tau_h(t, t'')$, if $t'' < t' < t$,
- iii. $\tau_h(t + kh, t' + kh) = \tau_h(t, t')$, for $k \in \mathbb{Z}$,
- iv. $\tau_h(t' + kh, t') = kh$, for $k \in \mathbb{Z}$,
- v. $|(t - t') - \tau_h(t, t')| \leq h$,

Proof. The first statement is a consequence of the inequality $0 \leq \alpha_{0h} \leq 1/b_0$. The additivity is immediate from the definition. The third statement is a consequence of the α_{0h} periodicity. Using that $\tau_h(t' + h, t') = h$, and the additivity property we have,

$$\tau_h(t' + kh, t') = \sum_{j=1}^k \tau_h(t' + jh, t' + (j-1)h) = kh.$$

For the last claim, we consider $t = t' + kh + sh$, with $k \in \mathbb{Z}$ and $0 \leq s < 1$, as $|1 - \alpha_{0h}(t)| \leq 1/b_0$, then

$$\begin{aligned} |(t - t') - \tau_h(t, t')| &= |kh + sh - (\tau_h(t' + kh + sh, t'))| \\ &= |(kh + sh) - \tau_h(t' + kh + sh, t' + kh) - \tau_h(t' + kh, t')| \\ &= |sh - \tau_h(t' + kh + sh, t' + kh)| \\ &= \left| \int_{t'+kh}^{t'+kh+sh} dt'' - \int_{t'+kh}^{t'+kh+sh} \alpha_{0h}(t'') dt'' \right| \\ &\leq \int_{t'+kh}^{t'+kh+sh} |1 - \alpha_{0h}(t'')| dt'' \leq h/b_0, \end{aligned}$$

that proves the last assertion. $\square$

We define $\Omega = \{(t, t') \in \mathbb{R}^2 : 0 \leq t' \leq t\}$ and the application $S_h : \Omega \rightarrow \mathcal{B}(X)$ defined by $S_h(t, t') = S(\tau_h(t, t'))$, from the previous lemma have:

Corollary 1. *Let $S : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$ a strongly continuous one-parameter semi-group of operators, we have that S_h satisfies:*

- i. $S_h(t, t) = \mathbb{I}$.
- ii. $S_h(t, t'') = S_h(t, t')S_h(t', t'')$, if $0 \leq t'' \leq t' \leq t$.
- iii. There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_h(t, t')\|_{\mathcal{B}(X)} \leq Me^{\omega(t-t')/b_0}$, for $(t, t') \in \Omega$.
- iv. If $u \in X$, The map $(t, t') \mapsto S_h(t, t')u$ is continuous.
- v. If $u \in D = \text{Dom}(A)$ and $t' \leq t \neq kh$ with $k \in \mathbb{Z}$, then the map $t \mapsto S_h(t, t')u$ is differentiable and we have

$$\partial_t S_h(t, t')u = \begin{cases} (1/b_0)AS_h(t, t')u & , \text{ if } kh \leq t \leq (k+1)h, \\ 0 & , \text{ if } (k+1)h \leq t \leq (k+2)h, \end{cases}$$

4. Approximate Solutions

In this section, we develop the main results (Proposition 4.2 and Theorem 4.2) that allow us to complete the splitting method. This enable us to obtain some properties of the solutions of problem (2.2) from the approximations obtained with the Lie-Trotter method. We define the system

$$\begin{cases} \partial_t u_h - \alpha_{0h}(t)Au_h = \sum_{j=1}^n \alpha_{jh}(t)F_j(t, x, u_h), \\ u_h = u_{h,0}, \end{cases} \quad (4.5)$$

with $\alpha_{0h}(t)$ as in (3), $u_h \in X$, $t > 0$, $F_j : \mathbb{R}_+ \times \mathbb{R}^d \times X \rightarrow X$ is a continuous function and X is a Banach space. We define α_j with $j = \{1, \dots, n\}$ as

$$\alpha_j(t) = \begin{cases} 1/(b_j - b_{j-1}), & \text{if } t \in [k + b_{j-1}, k + b_j], \\ 0 & , \text{if } t \notin [k + b_{j-1}, k + b_j], \end{cases} \quad (4.6)$$

for $k \in \mathbb{Z}$. Given $h > 0$, we define the function $\alpha_{jh} : \mathbb{R} \rightarrow \mathbb{R}$ as $\alpha_{jh}(t) = \alpha_j(t/h)$. Similarly we define the integral equation:

$$u_h(t) = S_h(t, 0)u_{h,0} + \int_0^t S_h(t, t') \sum_{j=1}^n \alpha_{jh}(t')F_j(t', x, u_h(t')) dt'. \quad (4.7)$$

Several well-known results can be extended to the approximate equation, for instance, local existence theorems.

Proposition 4.1. *Let $u_{h,0} \in C([0, T], \text{Dom}(A)) \cap \text{Lip}([0, T], X)$, if u_h is solution of the system (4.5) then u_h is solution of (4.7) for $t \in [0, T]$.*

Proof. The procedure is similar to [12], Lemma 4.1.1. $\square$

Theorem 4.1. *There exists a function $T^* : X \rightarrow \mathbb{R}_+$ such that for $u_{h,0} \in X$, exists a unique $u_h \in C([0, T^*(u_{h,0})], X)$ solution of (4.7) with $u_h(0) = u_{h,0}$. Moreover, one of the following alternatives holds:*

- $T^*(u_{h,0}) = \infty$;
- $T^*(u_{h,0}) < \infty$ and $\lim_{t \uparrow T^*(u_{h,0})} \|u_h(t)\|_X = \infty$.

Proof. The proof is similar to Theorem 4.3.4 in [12]. $\square$

In the following proposition, we show that the solution of the integral problem (4.7) corresponds to the approximations obtained with the Lie–Trotter method.

Proposition 4.2. *Let u_h the solution of (4.7), if $U_{n,k} = u_h(kh + h)$ y $V_{j,k} = u_h(kh + b_jh)$, then*

$$\begin{aligned} V_{0,k+1} &= S(h)U_{n,k}, \\ V_{1,k+1} &= N_1(kh + b_1h, kh + b_0h, V_{0,k+1}), \\ V_{2,k+1} &= N_2(kh + b_2h, kh + b_1h, V_{1,k+1}), \\ &\vdots \\ V_{n-1,k+1} &= N_{n-1}(kh + b_2h, kh + b_1h, V_{n-2,k+1}), \\ U_{n,k+1} &= N_n(kh + hb_n, kh + b_{n-1}h, V_{n-1,k+1}), \end{aligned} \tag{4.8}$$

where N_j is the flux associated to F_j , that is $w(t) = N_j(t, t_0, w_0)$ where w is the solution of

$$\begin{cases} \dot{w} = \alpha_{jh}(t)F_j(t, w(t)), \\ w(t_0) = w_0. \end{cases} \tag{4.9}$$

Proof. For $t_1 \in (0, t)$ it verifies

$$u_h(t) = S_h(t, t_1)u_h(t_1) + \int_{t_1}^t S_h(t, t') \sum_{i=1}^n \alpha_{ih}(t')F_i(t', x, u_h(t'))dt'$$

using that $t_1 = kh$ y $t = kh + b_0h$, we have

$$\begin{aligned} V_{0,k+1} &= S_h(kh + b_0h, kh)U_{n,k} \\ &+ \int_{kh}^{kh+b_0h} S_h(kh + b_0h, t') \sum_{i=1}^n \alpha_{ih}(t')F_i(t', x, u_h(t'))dt', \end{aligned}$$

given that $\alpha_{ih}(t) = 0$ for $t \in [kh, kh + b_0h)$, we have $\tau_h(kh + b_0h, kh) = h$ and, therefore, $V_{0,k+1} = S(h)U_{n,k}$. Similarly, $\alpha_{0h}(t) = 0$ for $t \in [kh + b_{(j-1)}h, kh +$

$b_j h$), then $\tau_h(t, kh + b_{(j-1)}h) = 0$ and, therefore,

$$u_h(t) = V_{j,k+1} + \int_{kh+b_{(j-1)}h}^t \sum_{i=1}^n \alpha_{ih}(t') F_i(t', x, u_h(t')) dt',$$

evaluating in $t = kh + b_j h$, we have that $\alpha_{ih}(t) = 0$ for $i \neq j$ and we obtain $V_{j,k+1} = N_1(kh + jh, kh + (j-1)h, V_{j-1,k+1})$. $\square$

Theorem 4.2. *Let $u \in C([0, T^*), X)$ the solution of the integral problem (2.2)*

$$u(t) = S(t)u_0 + \int_0^t S(t-t')F(t', x, u(t'))dt',$$

$T \in (0, T^*)$ $\forall \varepsilon > 0$. There exists $h^* > 0$ such that if $0 < h < h^*$, then u_h the solution of (4.7) with $u_{h,0} = u_0$, is defined in $[0, T]$ and verifies $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0, T]$.

To prove the theorem, we need two previous lemmas. We follow the procedure from Theorem 3.9 in [13] (see also [8]).

Lemma 4.1. *Let $f \in C([0, T], X)$, if*

$$I_h(t, t') = (S(t-t') - S_h(t, t'))f(t'),$$

then $\lim_{h \rightarrow 0^+} \sup_{(t,t') \in \Omega_T} \|I_h(t, t')\|_X = 0$, where $\Omega_T = \{(t, t') \in \mathbb{R}^2 : 0 \leq t' \leq t \leq T\}$.

Proof. Given $\varepsilon > 0$, there exists $g \in C([0, T], X)$ such that $g(t) \in \text{Dom}(A)$ for $t \in [0, T]$, $Ag \in C([0, T], X)$ and $\max_{t \in [0, T]} \|f(t) - g(t)\|_X < \varepsilon$.

$$\begin{aligned} \|(S(t-t') - S_h(t, t'))(f(t') - g(t'))\|_X &\leq 2Me^{\omega T/b_0} \max_{t \in [0, T]} \|f(t) - g(t)\|_X \\ &\leq 2Me^{\omega T/b_0} \varepsilon. \end{aligned} \quad (4.10)$$

On the other hand, we have

$$\begin{aligned} S(t-t')g(t') &= g(t') + \int_0^{t-t'} S(\xi) Ag(t') d\xi, \\ S_h(t, t')g(t') &= g(t') + \int_0^{\tau_h(t, t')} S(\xi) Ag(t') d\xi, \end{aligned}$$

subtracting both equations we obtain

$$(S(t-t') - S_h(t, t'))g(t') = \pm \int_{J(t, t')} S(\xi) Ag(t') d\xi,$$

where $J(t, t')$ is the interval $J(t, t') = [\min\{(t-t'), \tau_h(t, t')\}, \max\{(t-t'), \tau_h(t, t')\}]$, then

$$\begin{aligned} \|(S(t-t') - S_h(t, t'))g(t')\|_X &\leq Me^{\omega T/b_0} |(t-t') - \tau_h(t, t')| \max_{t \in [0, T]} \|Ag(t)\|_X \\ &\leq Me^{\omega T/b_0} h/b_0 \max_{t \in [0, T]} \|Ag(t)\|_X. \end{aligned} \quad (4.11)$$

From the Eqs. (4.10) and (4.11) we obtain the result. $\square$

Lemma 4.2. *Let $f \in C(\Omega_T, X)$, with Ω_T as in the previous lemma, if*

$$I_h(t) = \int_0^t (\alpha_{J,h}(t') - 1) f(t, t') dt',$$

then $\lim_{h \rightarrow 0^+} \sup_{t \in [0, T]} \|I_h(t)\|_X = 0$.

Proof. From the uniform continuity of f , we can see that there exists $\delta > 0$ such that if $0 \leq t', t'' \leq t \leq T$ and $|t' - t''| < \delta$, then $\|f(t, t') - f(t, t'')\|_X < \varepsilon$. Let $k = \lfloor t/h \rfloor$, we can write

$$I_h(t) = \sum_{j=1}^k \int_{(j-1)h}^{jh} (\alpha_{J,h}(t') - 1) f(t, t') dt' + \int_{kh}^t (\alpha_{J,h}(t') - 1) f(t, t') dt'.$$

As the mean value of $\alpha_{J,h}$ is 1 in intervals of length h , for $f_j \in X$ we have

$$0 = \int_{(j-1)h}^{jh} (\alpha_{J,h}(t') - 1) f_j dt', \quad (4.12)$$

therefore,

$$\int_{(j-1)h}^{jh} (\alpha_{J,h}(t') - 1) f(t, t') dt' = \int_{(j-1)h}^{jh} (\alpha_{J,h}(t') - 1) (f(t, t') - f_j) dt'.$$

Let $\Lambda = \{1, \dots, n\}$. If $h < \delta$ and $f_j = f(t, jh)$, then $\|f(t, t') - f_j\|_X < \varepsilon$ for $t' \in [(j-1)h, jh]$ and, therefore,

$$\left\| \int_{(j-1)h}^{jh} (\alpha_{J,h}(t') - 1) (f(t, t') - f_j) dt' \right\|_X \leq (\max_{J \in \Lambda} \alpha_J) \varepsilon h. \quad (4.13)$$

If $M = \max_{(t, t') \in \Omega_T} \|f(t, t')\|_X$, then we have

$$\begin{aligned} & \left\| \int_{kh}^t (\alpha_{J,h}(t') - 1) f(t, t') dt' \right\|_X \\ & \leq (\max_{J \in \Lambda} \alpha_J) \int_{kh}^t \|f(t, t')\|_X dt' \leq (\max_{J \in \Lambda} \alpha_J) M h. \end{aligned} \quad (4.14)$$

From (4.13), (4.12) y (4.14), we can obtain

$$\|I_h(t)\|_X \leq \sum_{j=1}^k (\max_{J \in \Lambda} \alpha_J) \varepsilon h + (\max_{J \in \Lambda} \alpha_J) M h \leq (\max_{J \in \Lambda} \alpha_J) T \varepsilon + (\max_{J \in \Lambda} \alpha_J) M h,$$

from where we get the result. $\square$

Proof (Proof of Theorem 4.2). If $[0, T_h^*)$ is the interval of existence of the integral equation (4.7), for $0 \leq t < \min\{T, T_h^*\}$ the subtraction $u(t) - u_h(t)$ satisfies

$$\begin{aligned} u(t) - u_h(t) &= (S(t) - S_h(t, 0)) u_0 + \int_0^t S(t-t') \sum_{j=1}^n F_j(t', x, u(t')) dt' \\ &\quad - \int_0^t S_h(t, t') \sum_{j=1}^n \alpha_{jh}(t) F_j(t', x, u_h(t')) dt'. \end{aligned}$$

If we define

$$\begin{aligned} I_{1,h}(t) &= (S(t) - S_h(t, 0)) u_0, \\ I_{2,h}(t) &= \sum_{j=1}^n \int_0^t \alpha_{jh}(t') (S(t - t') - S_h(t, t')) F_j(t', x, u(t')) dt', \\ I_{3,h}(t) &= \sum_{j=1}^n \int_0^t (1 - \alpha_{jh}(t')) S(t - t') F_j(t', x, u(t')) dt', \end{aligned}$$

then

$$\begin{aligned} u(t) - u_h(t) &= I_{1,h}(t) + I_{2,h}(t) + I_{3,h}(t) + \sum_{j=1}^n \int_0^t \alpha_{jh}(t') S_h(t, t') \\ &\quad \times (F_j(t', x, u(t')) - F_j(t', x, u_h(t'))) dt'. \end{aligned} \quad (4.15)$$

Using the Lemma 4.1, using $f(t) = u_0$, we can see that $\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \|I_{1,h}(t)\|_X = 0$. Given that,

$$\begin{aligned} \|I_{2,h}(t)\|_X &\leq \sum_{j=1}^n \int_0^t \| (S(t - t') - S_h(t, t')) F_j(t', x, u(t')) \|_X dt' \\ &\leq T \sum_{j=1}^n \sup_{(t, t') \in \Omega_T} \| (S(t - t') - S_h(t, t')) F_j(t', x, u(t')) \|_X, \end{aligned}$$

using once again Lemma 4.1 for $f(t) = F(t, x, u(t))$, we obtain $\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \|I_{2,h}(t)\|_X = 0$. The map $f(t, t') = S(t - t') F(t', x, u(t'))$ is continuous in Ω_T , therefore, from Lemma 4.2, we can deduce $\lim_{h \rightarrow 0} \sup_{t \in [0, T]} \|I_{3,h}(t)\|_X = 0$.

We consider $R = \max_{t \in [0, T]} \|u(t)\|_X + \varepsilon$ and L_j the Lipschitz constant of F_j for $B_R(0) \subset X$, and $\sum_{j=1}^n L_j = L$, if we define

$$J_\varepsilon = \{0 \leq t < \min\{T, T_h^*\} : \|u_h(t')\|_X < R + \varepsilon, 0 \leq t' \leq t\},$$

from the estimate (4.15) we obtain for $t \in J_\varepsilon$:

$$\begin{aligned} \|u(t) - u_h(t)\|_X &\leq \|I_{1,h}(t)\|_X + \|I_{2,h}(t)\|_X + \|I_{3,h}(t)\|_X \\ &\quad + M e^{\omega T/b_0} L \int_0^t \|u(t') - u_h(t')\|_X dt', \end{aligned}$$

and from Gronwall's Lemma

$$\|u(t) - u_h(t)\|_X \leq e^{CT} \left(\sup_{t \in [0, T]} \|I_{1,h}(t)\|_X + \sup_{t \in [0, T]} \|I_{2,h}(t)\|_X + \sup_{t \in [0, T]} \|I_{3,h}(t)\|_X \right)$$

where $C = M e^{\omega T/b_0} L$. Taking $h^* > 0$ small enough, we have that $\|u(t) - u_h(t)\|_X < \varepsilon$ for $t \in J_\varepsilon$ and $0 < h < h^*$. Then we have that

$$\|u_h(t)\|_X \leq \|u(t)\|_X + \|u(t) - u_h(t)\|_X \leq \max_{0 \leq t \leq T} \|u(t)\|_X + \varepsilon = R \quad (4.16)$$

for every $t \in J_\varepsilon$.

Suppose that $T_h^* < T$. Let $T_\varepsilon = \sup J_\varepsilon$ and $T_\varepsilon < T_h^* < T$. Using inequality (4.16) we have that $\|u_h(T_\varepsilon)\|_X < R + \varepsilon$. As u_h is continuous, there exists $T_1 \in (T_\varepsilon, T_h^*)$ such that $\max_{0 \leq t \leq T_1} \|u_h(t)\|_X < R + \varepsilon$. Therefore, $T_1 \in J_\varepsilon$ and $T_1 > T_\varepsilon$. This means that $T_\varepsilon \neq \sup J_\varepsilon$, then it must be that, $T_\varepsilon = T_h^*$. By definition of J_ε , we have that $\|u_h(t)\|_X < R + \varepsilon$ for $0 \leq t < T_h^*$. But on the other hand, we have that $\lim_{t \rightarrow T_h^*} \|u_h(t)\| = \infty$, a contradiction.

Therefore, $T_h^* \neq T_\varepsilon$ and $T = \sup J_\varepsilon$, with $T \leq T_h^*$ and also $\|u(t) - u_h(t)\|_X \leq \varepsilon$ for $t \in [0, T]$ what proves the Theorem. $\square$

5. Fractional Heat Equation in Almost Periodic Spaces

We consider the following system,

$$\begin{cases} \partial_t u = \sigma(-\partial_{xx})^\beta u + |u|^2 u + V(x)u, \\ u(0) = u_0 \end{cases} \quad (5.17)$$

where $u(x, t)$ is a complex valued function for $x \in \mathbb{R}$, $t > 0$, and $V(x)$ is a continuous map. We define the set of almost periodic functions [7] as,

$$P(\mathbb{R}) = \{u \in C_u(\mathbb{R}) : u = \sum_{j=1}^{\infty} a_j e^{ix\lambda_j} : \lambda_j \in \mathbb{R}\}$$

equipped with the uniform continuous norm.

We use the method developed in Sect. 4 to prove well-posedness of problem (5.17), for the following subspace of almost periodic functions,

$$A_\Lambda(\mathbb{R}) = \{u \in C_u(\mathbb{R}) : u = \sum_{\lambda_j \in \Lambda} a_{\lambda_j} e^{i\lambda_j x}; \sum_{\lambda_j \in \Lambda} |a_{\lambda_j}| < \infty\} \quad (5.18)$$

where $\Lambda \subset \mathbb{R}$ is a enumerable set, and if $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ then $\lambda_1 - \lambda_2 + \lambda_3 \in \Lambda$.

We define the norm:

$$\|u\|_{A_\Lambda} = \sum_{\lambda_j \in \Lambda} |a_{\lambda_j}| \quad (5.19)$$

Remark 5.1. Λ is an empty set or a coset of a countable additive subgroup of $\mathbb{R}$. Suppose $\Lambda \neq \emptyset$. The set of differences of elements of Λ forms a countable additive subgroup of $\mathbb{R}$, as if $u = t - s$ and $v = r - w$ with $r, s, t, w \in \Lambda$, then $u + v = (t - s + r) - w$.

Let G denote this subgroup and $t \in \Lambda$. Then if $u \in G$ we have $u = s - r$ with $s, r \in \Lambda$ so $t + u = t + s - r \in \Lambda$. Conversely, if $r \in \Lambda$ then $r = t + u$, where $u = r - t \in G$. Thus $\Lambda = t + G$.

Remark 5.2. Without loss of generality, from now on, Λ is an infinite numerable set.

Lemma 5.1. $A_\Lambda(\mathbb{R})$ is a Banach space.

Proof. Let $\{u_n\}_{n \in \mathbb{N}}$ a Cauchy sequence such that $u_n \in A_\Lambda(\mathbb{R}) \subset P(\mathbb{R})$. Then $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence, that is, $\|u_n\|_{A_\Lambda} = \sum_{j=1}^{\infty} |a_j^n| \leq K$ for all $n \in \mathbb{N}$. Additionally, $\{u_n\}_{n \in \mathbb{N}}$ is a convergent sequence in $P(\mathbb{R})$, i.e. $u_n \rightarrow u$ with $u \in P(\mathbb{R})$ and $\|u\|_{A_\Lambda} = \sum_{j=1}^N |a_j|$. Therefore, we have for $N \in \mathbb{N}$:

$$\sum_{j=1}^N |a_j| = \lim_{n \rightarrow \infty} \sum_{j=1}^N |a_j^n| \leq K \implies \lim_{N \rightarrow \infty} \sum_{j=1}^N |a_j| = \sum_{j=1}^{\infty} |a_j| \leq K$$

On the other hand, we can define for $u, v \in P(\mathbb{R})$:

$$\langle u(x)v(x) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(x)v(x)dx \quad (5.20)$$

This limit exists and in particular, we have a normalized orthogonal system in the following sense (see [7]):

$$\langle e^{i\lambda_1 x} e^{-i\lambda_2 x} \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{i\lambda_1 x} e^{-i\lambda_2 x} = \begin{cases} 0 & \text{for } \lambda_1 \neq \lambda_2 \\ 1 & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

We define $a(\lambda) = \langle u(x)e^{-i\lambda x} \rangle$. It is proved in [7] that for $u \in P(\mathbb{R})$, $u = \sum_{j=1}^{\infty} a_j e^{i\lambda_j x}$, we have,

$$\langle u(x)e^{-i\lambda_j x} \rangle = a(\lambda_j) = a_j \quad (5.21)$$

On the other hand, if μ is not a phase of $u(x)$ then $a(\mu) = 0$. If $\{u_n\}_{n \in \mathbb{N}} \in A_\Lambda(\mathbb{R})$ is a Cauchy sequence, then

$$a_j^n = \langle u_n(x)e^{-i\lambda_j x} \rangle \xrightarrow{n \rightarrow \infty} \langle u(x)e^{-i\lambda_j x} \rangle = a_j$$

where $\lambda_j \in \Lambda$. As (5.18) is met then $u \in A_\Lambda(\mathbb{R})$. $\square$

We denote $T(t)$ as the one parameter semigroup that solves the underlying linear equation

$$\begin{cases} \partial_t u = \sigma(-\partial_{xx})^\beta u \\ u(0) = u_0 \end{cases}$$

Lemma 5.2. Let $0 < \beta \leq 1$ and $g_\beta \in C_0(\mathbb{R}^d)$ such that $\hat{g}_\beta(\gamma) = e^{-|\gamma|^{2\beta}}$, it holds g_β is positive, invariant under rotations of $\mathbb{R}^d$, integrable and

$$\int_{\mathbb{R}} g_\beta(x)dx = 1.$$

Proof. The first statement follows from Theorem 14.14 of [14], the remaining claims are immediate from the definition of $\hat{g}_\beta$. $\square$

Proposition 5.1. Let $\sigma > 0$ and $0 < \beta \leq 1$, the function $G_{\sigma,\beta}$ given by

$$G_{\sigma,\beta}(x, t) = (\sigma t)^{-\frac{1}{2\beta}} g_\beta((\sigma t)^{-\frac{1}{2\beta}} x),$$

for any $t > 0$, it holds:

$$1. \ G_{\sigma,\beta}(\cdot, t) > 0;$$

2. $G_{\sigma,\beta}(\cdot, t) \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} G_{\sigma,\beta}(x, t) dx = 1;$$

3. $\partial_t G_{\sigma,\beta} + \alpha(-\partial_{xx})^\beta G_{\sigma,\beta} = 0$ for $t > 0$;

4. $G_{\sigma,\beta}(\cdot, t) * G_{\sigma,\beta}(\cdot, t') = G_{\sigma,\beta}(\cdot, t + t')$, for $t, t' > 0$.

Proof. The first and second statements are a consequence of the definition of $\hat{g}_\beta$. The third and fourth statements are immediate applying Fourier transform. $\square$

Lemma 5.3. *If $u_0 \in P(\mathbb{R})$ then $T(t)u_0 = G_{\sigma,\beta}(\cdot, t) * u_0 \in P(\mathbb{R})$ for $t > 0$.*

Proof. We recall the equivalent definition of Almost periodic functions from [7]. If $u_0 \in P(\mathbb{R})$ then given $\epsilon > 0$ there exists a relative dense set of translation numbers $\tau(\epsilon)$ such that $|u_0(x + \tau) - u_0(x)| < \epsilon$. Then we have,

$$\begin{aligned} |T(t)u_0(x + \tau) - T(t)u_0(x)| &\leq \int_{\mathbb{R}} |G_{\sigma,\beta}(y, t) u_0(x + \tau - y) - G_{\sigma,\beta}(y, t) u_0(x - y)| dy \\ &= \int_{\mathbb{R}} |G_{\sigma,\beta}(y, t)| |(u_0(x - y + \tau) - u_0(x - y))| dy \end{aligned}$$

As $u_0 \in P(\mathbb{R})$ and Proposition 5.1 we have that $|T(t)u_0(x + \tau) - T(t)u_0(x)| < \epsilon$ and $T(t)u_0 \in P(\mathbb{R})$. $\square$

Lemma 5.4. *If $u_0 \in A_\Lambda(\mathbb{R})$ then $T(t)u_0 = G_{\sigma,\beta}(\cdot, t) * u_0 \in A_\Lambda(\mathbb{R})$ for $t > 0$.*

Proof. By Lemma 5.3, we have that the limit 5.20 exists. We first consider the following integral,

$$\begin{aligned} &\frac{1}{L} \int_0^L (G_{\sigma,\beta}(\cdot, t) * u_0) e^{-i\xi \cdot x} dx \\ &= \frac{1}{L} \int_0^L \int_{\mathbb{R}} G_{\sigma,\beta}(y, t) u_0(x - y) e^{-i\xi \cdot (x - y)} e^{-i\xi \cdot y} dy dx \\ &= \int_{\mathbb{R}} G_{\sigma,\beta}(y, t) e^{-i\xi \cdot y} \left(\frac{1}{L} \int_0^L u_0(x - y) e^{-i\xi \cdot (x - y)} dx \right) dy \\ &= \int_{\mathbb{R}} G_{\sigma,\beta}(y, t) e^{-i\xi \cdot y} \left(\frac{1}{L} \int_{-y}^{-y+L} u_0(x) e^{-i\xi \cdot x} dx \right) dy \end{aligned}$$

We define $h(y, L)$ as,

$$h(y, L) = \frac{1}{L} \int_{-y}^{-y+L} u_0(x) e^{-i\xi \cdot x} dx$$

$h(y, L)$ is bounded and verifies $\lim_{L \rightarrow \infty} h(y, L) = \langle u_0 e^{-i\xi \cdot y} \rangle$, by dominated convergence theorem we have,

$$\begin{aligned} \langle (T(t)u_0) e^{-i\xi \cdot y} \rangle &= \langle (G_{\sigma,\beta}(\cdot, t) * u_0) e^{-i\xi \cdot y} \rangle = \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L (G_{\sigma,\beta}(\cdot, t) * u_0) e^{-i\xi \cdot x} dx \\ &= \int_{\mathbb{R}} G_{\sigma,\beta}(y, t) e^{-i\xi \cdot y} \langle u_0 e^{-i\xi \cdot y} \rangle dy = \hat{G}_{\sigma,\beta}(\xi, t) \langle u_0 e^{-i\xi \cdot y} \rangle. \end{aligned}$$

Where $\langle u_0 e^{-i\xi \cdot y} \rangle$ is the ξ -phase coefficient of u_0 as defined in 5.21. Then if a_{ξ_j} are the phase coefficients of u_0 then $\hat{G}_{\sigma,\beta}(\xi, t) a_{\xi_j}$ are the phase coefficients of $T(t)u_0$. Therefore, $T(t)u_0$ maintains the same phases as u_0 and considering that $\hat{G}_{\sigma,\beta}(\xi, t) \in L^\infty(\mathbb{R})$ then $\sum_{j=1}^\infty |\hat{G}_{\sigma,\beta}(\xi, t) a_{\xi_j}| < \infty$ and $T(t)u_0 \in A_\Lambda(\mathbb{R})$. $\square$

The following Lemma proves well-posedness for the nonlinear equation,

$$\begin{cases} \partial_t z = |z|^2 z \\ z(0) = z_0 \end{cases} \quad (5.22)$$

Lemma 5.5. *The map $F(u) = |u|^2 u$ is locally Lipschitz and $F : A_\Lambda \rightarrow A_\Lambda$.*

Proof. We consider, $u, v, w \in A_\Lambda(\mathbb{R})$, where

$$u = \sum_{j=1}^\infty a_j e^{ix(\lambda_j)}, \quad v = \sum_{j=1}^\infty b_j e^{ix(\lambda_j)}, \quad w = \sum_{j=1}^\infty c_j e^{ix(\lambda_j)}$$

If $d_{kjl} = a_k \bar{b}_j c_l$ then,

$$\|u\bar{v}w\|_{A_\Lambda} = \sum_{k,j,l=1}^\infty |d_{kjl}| = \sum_{k=1}^\infty \left| \sum_{j=1}^\infty \sum_{l=1}^\infty a_k \bar{b}_j c_l \right| \leq \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{l=1}^\infty |a_k| |\bar{b}_j| |c_l| < \infty$$

Then, we have that, $\|u\bar{v}w\|_{A_\Lambda} \leq \|u\|_{A_\Lambda} \|v\|_{A_\Lambda} \|w\|_{A_\Lambda}$.

Specifically, we have for $u = \sum_{j=1}^\infty a_j e^{ix(\lambda_j)}$. Then

$$|u|^2 u = \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{l=1}^\infty a_k \bar{a}_j a_l e^{ix(\lambda_k - \lambda_j + \lambda_l)} \quad (5.23)$$

Then $\lambda_k - \lambda_j + \lambda_l = \lambda_{kjl} \in \Lambda$ and $b_{kjl} = a_k \bar{a}_j a_l$.

We can see that,

$$\sum_{k,j,l=1}^\infty |b_{kjl}| = \sum_{k=1}^\infty \left| \sum_{j=1}^\infty \sum_{l=1}^\infty a_k \bar{a}_j a_l \right| \leq \sum_{k=1}^\infty \sum_{j=1}^\infty \sum_{l=1}^\infty |a_k| |\bar{a}_j| |a_l| = \left(\sum_{m=1}^\infty |a_m| \right)^3$$

as $\sum_{m=1}^\infty |a_m| < \infty$ then $\sum_{m=1}^\infty |b_m| < \infty$. Now, we prove that F is a locally Lipschitz map in $A_\Lambda(\mathbb{R})$. Let $u, v \in A_\Lambda(\mathbb{R})$,

$$\begin{aligned} \|F(u) - F(v)\|_{A_\Lambda} &= \left\| |u|^2 u - |v|^2 v \right\|_{A_\Lambda} \leq \left\| |u|^2 u - |u|^2 v + |u|^2 v - |v|^2 u + |v|^2 u - |v|^2 v \right\|_{A_\Lambda} \\ &= \|u\bar{u}(u-v) + uv(\bar{u}-\bar{v}) + \bar{v}v(u-v)\|_{A_\Lambda} \leq \|u\bar{u}(u-v)\|_{A_\Lambda} + \|uv(\bar{u}-\bar{v})\|_{A_\Lambda} + \|\bar{v}v(u-v)\|_{A_\Lambda} \\ &\leq \|u\|_{A_\Lambda} \|u\|_{A_\Lambda} \|(u-v)\|_{A_\Lambda} \\ &\quad + \|u\|_{A_\Lambda} \|v\|_{A_\Lambda} \|(\bar{u}-\bar{v})\|_{A_\Lambda} \\ &\quad + \|\bar{v}\|_{A_\Lambda} \|v\|_{A_\Lambda} \|(u-v)\|_{A_\Lambda} \end{aligned}$$

We used that $\|u\|_{A_\Lambda} = \|\bar{u}\|_{A_\Lambda}$. Then,

$$\|F(u) - F(v)\|_{A_\Lambda} = \left\| |u|^2 u - |v|^2 v \right\|_{A_\Lambda} \leq K \|u - v\|_{A_\Lambda}.$$

$\square$

We now analyze the non-linear problem associated with the potential term, that is

$$\begin{cases} \partial_t z = V(x)z, \\ z(0) = z_0, \end{cases} \quad (5.24)$$

We consider an almost periodic potential, with a specific real phase set, not necessarily the same phase as u_0 . Let Λ and G as in Remark 5.1. We allow the potential to have *complementary* phases, i.e. $V(x) \in A_G(\mathbb{R})$.

Lemma 5.6. *If $\lambda \in \Lambda$ and $g \in G$ then $\lambda + g \in \Lambda$*

Proof. See Remark 5.1. □

Lemma 5.7. *If $V(x) \in A_G(\mathbb{R})$ and $u \in A_\Lambda(\mathbb{R})$ then $F(u) = V(x)u$ is a bounded locally Lipschitz map and $F : A_\Lambda(\mathbb{R}) \rightarrow A_\Lambda(\mathbb{R})$.*

Proof. Let $u = \sum_{j=1}^{\infty} a_j e^{ix\lambda_j}$ and $V(x) = \sum_{k=1}^{\infty} b_k e^{ixg_k}$. Then,

$$V(x)u = \sum_{j,k=1}^{\infty} c_{jk} e^{ix(\lambda_j + g_k)}$$

with $c_{jk} = a_j b_k$ and by Lemma 5.6 we have that, $\lambda_j + g_k \in \Lambda$. As $\sum_{j,k=1}^{\infty} |c_{jk}| < \infty$, then $V(x)u \in A_\Lambda(\mathbb{R})$. Additionally, F is a bounded map, noting that $\|V(x)u\|_{A_\Lambda} = \|V(x)\|_{A_G} \|u\|_{A_\Lambda}$. On the other hand, we have that,

$\|F(u) - F(v)\|_{A_\Lambda} = \|V(x)(u - v)\|_{A_\Lambda} \leq \|V(x)\|_{A_G} \|u - v\|_{A_\Lambda} \leq K \|u - v\|_{A_\Lambda}$ then F is a bounded locally Lipschitz map. □

We now join all the previous results and use the splitting method to prove well-posedness of equation (5.17) in $A_\Lambda(\mathbb{R})$. We denote by $\mathbf{N}_1 : \mathbb{R} \times \mathbb{R} \times C_u(\mathbb{R}) \rightarrow C_u(\mathbb{R})$ the flow generated by the equation (5.22) as $u(t) = \mathbf{N}_1(t, t_0, u_0)$, and similarly $u(t) = \mathbf{N}_2(t, t_0, u_0)$ the flow generated by the equation (5.24) defined for $t_0 \leq t < t_0 + T^*(t_0, u_0)$.

Theorem 5.1. *If $u_0 \in A_\Lambda(\mathbb{R})$, and u is the solution of problem 5.17 then $u(t) \in A_\Lambda(\mathbb{R})$ for all $t \in (0, T^*(u_0))$.*

Proof. For $t \in [0, T^*(u_0))$, let $n \in \mathbb{N}$, $h = t/n$ and $\{U_{2,k}\}_{0 \leq k \leq n}, \{V_{j,k}\}_{1 \leq k \leq n}$ be the sequences given by $U_{2,0} = u_0$,

$$\begin{aligned} V_{0,k+1} &= T(h)U_{2,k}, \\ V_{1,k+1} &= \mathbf{N}_1(kh + 2h/3, kh + h/3, V_{0,k+1}), \\ U_{2,k+1} &= \mathbf{N}_2(kh + h, kh + 2h/3, V_{1,k+1}), \quad k = 0, \dots, n. \end{aligned}$$

We claim that $U_{2,k+1} \in A_\Lambda(\mathbb{R})$ for $k = 0, \dots, n$. Clearly, the assertion is true for $k = 0$. If $U_{2,k} \in A_\Lambda(\mathbb{R})$, then using Lemma 5.4, we can see that

$$V_{0,k} = T(h)U_{2,k} \in A_\Lambda(\mathbb{R}).$$

from Lemma 5.5, we have $V_{1,k} = \mathbf{N}_1(kh + 2h/3, kh + h/3, V_{0,k}) \in A_\Lambda(\mathbb{R})$ and from Lemma 5.7 we have $U_{2,k+1} = \mathbf{N}_2(kh + h, kh + 2h/3, V_{1,k}) \in A_\Lambda(\mathbb{R})$. We now recall Theorem 4.2 that assures us that $U_{2,n} \rightarrow u(t)$ when $n \rightarrow \infty$. As $A_\Lambda(\mathbb{R})$ is a Banach space, we obtain the result. □

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